A New Approach for the Equation of State of Nucleonic Matter from the Quantum Second Virial Coefficient within the Framework of the “Bethe Homework Problem”

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Abstract

The whole idea of the present approach rests on determining first the quantum second virial coefficient $B_q$. This is accomplished from ‘first principles’ for nonrelativistic nucleonic matter in the temperature-range 5-60 MeV and density-range 0.05-0.17 MeV for relatively low energies $\leq$ 300 MeV. In the present preliminary paper, this idea is tested out for the “Bethe homework problem”, where the input potential is the exceedingly simple $^1S_0$ soft-core Reid-93 potential. The many-body phase shifts are determined within the framework of a generalized scattering theory (the Galitskii-Migdal-Feynman formalism). These medium phase shifts are then inserted in the Bethe-Uhlenbeck formula to determine $B_q$. It turns out that the positive values of $B_q$ represent an overall repulsive interaction; whereas the negative values represent an overall attractive interaction. $B_q$ is then used to investigate the equation of state (pressure-volume-temperature relations) of the system. Other thermophysical properties – including Helmholtz free energy, entropy, total internal energy, specific heat capacity, and chemical potential – are also determined.

Keywords: Equation of state; Nuclear matter; Neutron matter; Quantum second virial coefficient; Bethe homework problem.

1. Introduction

Nuclear matter is an idealized infinite system consisting of a very large number of protons and neutrons. The Coulomb interaction is switched off. The significance of this system stems from the fact that it can be regarded as a first approximation to heavy nuclei.
as well as from the role it plays in astrophysics (neutron-star cooling, Steller collapse, supernova explosions, and heavy-ion physics) (Greiner and Stocker, 1985, Lattimer and Prakash, 2000; Steiner et al., 2005; Gandolfi, 2015; Haensel and Zdunik, 2016)). In the nonrelativistic, low-energy (≤300 MeV) regime, the nucleons interact by the well-known nucleon-nucleon (NN) potential (Engvik et al., 1997; Walecka, 2004).

Defining R as the ratio of the number of protons Z to that of the neutrons N, then for R = 1, the system will be symmetric nuclear matter (N = Z); for R = 0, the system will be pure neutron matter (Z = 0). Alternatively, the parameter \( \alpha = (1-R)/(1+R) = (N-Z)/(N+Z) \) is zero for symmetric nuclear matter and 1 for pure neutron matter (Braghin, 2003; Oyamatsu and Iida, 2004). Both these cases will be studied in the present work. The focus here is on nucleonic matter in the region where it behaves as a nonideal gas. The following are some figures of interest: the equilibrium density of nuclear matter is 0.17 nucleons/F³, the interparticle spacing is 1.12 F, the Fermi momentum is 1.36 F⁻¹, and the equilibrium binding energy per nucleon is 15.7 MeV (Bethe, 1971; Lattimer and Prakash, 2000).

There have been several approaches for determining the equation of state and other properties of nucleonic matter (Müther and Polls, 2000). Examples are: the diagrammatic techniques of Brueckner-Bethe-Goldstone (BBG) (Day, 1967) and Galitskii-Migdal- Feynman (GMF) (Fetter and Walecka, 1971; Ghassib et al., 1976; Bishop et al., 1976), the Brueckner-Hartree-Fock approach (Li et al., 2002; Hassaneen et al., 2011; Hu et al., 2016), the quantum Monte-Carlo methods (Ceperley et al., 1977; Modarres and Mosfegh, 2004), and the static fluctuation approximation (SFA) (Ghulam et al., 2007; 2008).

The present approach starts from a new point of departure – namely, the quantum second virial coefficient \( B_q \). Specifically, the following scheme is adopted: First, the medium phase shifts of the input potential are determined. For this purpose, the Galitskii-Migdal-Feynman (GMF) T-matrix integral equation is invoked. This is a Dyson-like equation (Fetter and Walecka, 1971) – just as its zero-density limit, namely,
the Lippmann-Schwinger (LS) t-matrix (Joachain, 1979). Then, these phase shifts are fed into the Beth-Uhlenbeck formula (Uhlenbeck and Beth, 1936; Beth and Uhlenbeck, 1937) which gives $B_q$. Finally, other thermophysical properties of the system are determined from this coefficient, with special emphasis on the equation of state. In this work, our Dyson-like equations, which are of the Fredholm type (Mathews and Walker, 1969), are solved quite accurately using a powerful matrix inversion technique (Ghassib et al., 1976; Bishop et al., 1977).

In order to debug our codes and test the methodologies involved in our approach, we use the Bethe-homework problem (Najwa et al., 2008). This was originally suggested by Bethe for symmetric nuclear matter at zero temperature so as to sort out the confusion created by so many input potentials and methods. With the sole choice of the $^1S_0$-channel of the Reid-68 soft-core potential (Reid, 1968), one could then compare meaningfully the various methods available. Some 35 years later, Ghassib and co-workers (Najwa et al., 2008) extended the ‘model’ to finite temperatures, this time with the Reid-93 $^1S_0$ potential (Stoks et al., 1994) so as to introduce the SFA approach in nucleonic matter. Likewise, we are using here this model to introduce the present approach that is based on $B_q$.

This paper is organized as follows. Our theoretical framework just outlined is elaborated in Section 2. Next, in Section 3, the results of our calculations are presented and discussed. Finally, in Section 4, the paper is concluded with some general remarks.

2. Theoretical Framework

We consider a system of $N$ interacting nucleons, mass $m$ each, occupying a (normalization) volume $\Omega$. As usual for such many-body systems, the thermodynamic limit is considered, namely, $N \to \infty$, $\Omega \to \infty$; but the ratio $n = N/\Omega$, which is simply the number density, remains constant. Throughout this work, the natural system of units is used, such that $m = \hbar = k_B = 1$, $k_B$ being Boltzmann’s constant. The conversion factor is $\hbar^2/m = 41.461$ MeV/F$^2$. All physical quantities can then be expressed in terms of length $L$: e.g., [linear momentum] $\equiv [L^{-1}]$, and [energy] $\equiv [L^{-2}]$. The nucleons interact through the well-known NN potential (Reid, 1968; Holinde, 1980; Stoks et al., 1994; Hassaneen et al., 2008).
al., 2011; Naghdi, 2014). However, as already mentioned, a simple model potential is used in this work, as will be elaborated presently.

2.1 A model NN interaction: Bethe homework potential

The NN interaction is of fundamental importance in nuclear physics; it is the starting point for any microscopic understanding of nuclear properties (Holinde, 1980). There are quite a few phenomenological NN potentials that fit the available scattering phase shifts. Most of these are static potentials depending on the internucleon separation $r$ as well as the angular momentum, spin, and other quantum numbers of the two interacting nucleons (Hassaneen et al., 2011; Naghdi, 2014).

In the original Bethe-homework problem, the input binary potential $V(r)$ [in MeV] was chosen as the $^1S_0$-channel of the Reid-68 potential, given by

$$V(r) = 9263.1 e^{-4.9r}/r.$$  
(1)

This is purely repulsive. In addition, the calculations were performed at absolute zero-temperature.

On the other hand, in this work, our input potential is the $^1S_0$-channel of Reid-93 (Stoks et al., 1994):

$$V(x) = m\pi(0.1756084y(2,x) - 14.4234y(3,x) + 151.8489y(4,x) - 686.823y(5,x) + 1104.157y(6,x),$$  
(2)

where

if $x < 1 \times 10^{-4}$, then $y(i,x) = -i + 4 + i^2/16$;

if $x > 1 \times 10^{-4}$, then $y(i,x) = \exp(-i x)/x - (\exp(-8x)/x)(1 + (8^2 - i^2)x/16)$;

$x \equiv \mu r$; $\mu \equiv m\pi/\hbar c = 0.7F^{-1}$, $m\pi$ is the pion mass $[(m\pi^* + 2m\pi)/3 = 138.0363$ MeV], $\hbar$ is Dirac’s constant ($\equiv \hbar/2\pi$), and $c$ is the speed of light in vacuum. This potential has some
of the overall characteristics of the general NN potential: It exhibits both the short-range strongly repulsive, and the long-range attractive OPEP, regions, together with the minimum in between representing the equilibrium configuration. Figure 1 shows both “Bethe-homework potentials” as functions of $r$.

![Graph showing the original Bethe-homework potential and our present model potential as functions of the internucleon separation $r$](image)

Figure 1: The original Bethe-homework potential [i.e., the Reid-68 soft-core $^1S_0$ potential], and our present model potential [the Reid-93 soft-core $^1S_0$ potential], as functions of the internucleon separation $r$ [F].

2.2 A generalized scattering theory:

2.2.1 In free space: Lippmann-Schwinger (LS) equation

As a first step, the two-nucleon problem in free space is considered. This is described by the LS equation, which is simply Schrödinger’s equation in momentum space; it is also valid for the medium, albeit in the zero-density limit only. The
scattering-state solution is represented by the LS t-matrix (Bishop et al., 1977). This is given by a Dyson-like integral equation of the Fredholm type (Mathews and Walker, 1969) which incorporates multiple scattering between the interacting nucleons (Joachain, 1979; Sakurai, 1987). Operationally, this is given by the equation

\[ t = V - V \frac{1}{e} t, \]  

where \( V \) is the internucleon potential and \( e \) is the difference between the relative energy of the outgoing particles and that of the incoming pair.

The corresponding momentum representation is

\[ \langle \hat{p}|t|\hat{p} \rangle = \langle \hat{p}|V|\hat{p} \rangle - \langle \hat{p}|V \frac{1}{e} |\hat{p}' \rangle \langle \hat{p}'|t|\hat{p} \rangle; \]  

or, more explicitly,

\[ t(\hat{p}, \hat{p}'; s, \hat{P}) = \frac{2m_r}{\hbar^2} V(\hat{p}, \hat{p}') - (2\pi)^{-3} \int d\hat{k} \frac{2m_r}{\hbar^2} \frac{V(\hat{p}, \hat{k})}{k^2 - s - i\eta} \times t(\hat{k}, \hat{p}'; s, \hat{P}) \].  

Here: \( \hat{p} \) and \( \hat{p}' \) are the relative incoming and outgoing momenta; \( \hat{P} \) is the center-of-mass momentum; \( m_r \) is the reduced mass of the interacting pair = \( \frac{1}{2} m \); the parameter \( s \) is the total energy of the interacting pair in the center-of-mass frame and is given by

\[ s = \frac{2m_r}{\hbar^2} \left( P_0 - \frac{\hbar^2 P^2}{2M} \right), \]  

where \( P_0 \) is the total energy of the pair: \( P_0 = \frac{\hbar^2 P^2}{2M} + \frac{\hbar^2 k^2}{2m_r}, P^2 \) being the energy carried by the center of mass, and \( M = 2m \); so that

\[ s \equiv k^2; \]  

\[ u \equiv \frac{2m_r V}{\hbar^2} \equiv V, \]  

\( V \) being the Fourier-Bessel transform of the input potential.

The angular dependence of \( t \) and \( V \) is ‘processed’ by the well-known technique of partial-wave decomposition (Ghassib et al., 1976) This is straightforward, thanks to
the translational invariance characterizing the present infinite and homogenous system. The implication is that the single-particle wave functions are plane waves, represented by $\Omega^{-\frac{1}{2}}e^{ik\cdot r}$ (Day, 1967). For central potentials, the familiar Euler expansion can be used (Mathews and Walker, 1969):

$$e^{ik\cdot r} = 4\pi \sum \epsilon' j_{\epsilon}(kr)Y_{\epsilon'}(\hat{k})Y_{\epsilon'}(\hat{r}),$$

(7)

where $j_{\epsilon}(kr)$ is the spherical Bessel function of order $\ell$ and argument $kr$, and $Y_{\epsilon}(\hat{k})$ is the spherical harmonic of order $\ell$ and appropriate angular arguments.

With

$$V(\hat{k}, \hat{k}') = \int d^3r \, V(r) e^{iq\cdot r}; \quad \bar{q} = \hat{k} - \hat{k}',$$

(8)

and the identity $P_{\ell} (\hat{p}, \hat{p}') = \frac{4\pi}{2\ell + 1} \sum_m Y_{\ell m}^* (\hat{p}') Y_{\ell m} (\hat{p})$, $P_{\ell} (\hat{p}, \hat{p}')$ being the Legendre polynomial of the first kind (of order $\ell$ and argument $\hat{p}, \hat{p}'$), then substituting the Euler expansion, Eq. (7), into (8), one obtains

$$V(\hat{p}, \hat{p}') = \sum_{\ell} (2\ell + 1)V_{\ell} (p, p') P_{\ell} (\hat{p}, \hat{p}'),$$

(9)

where

$$V_{\ell} (p, p') = 4\pi \int_0^\infty dk \, k^2 j_{\ell} (pr) V(r) j_{\ell} (p'r).$$

(10)

By the same token,
\[ t(\vec{p}, \vec{p}'; s, P) = \sum_{\ell} (2\ell + 1) t_{\ell}(p, p'; s, P) \mathcal{P}_\ell(\hat{p}, \hat{p}'). \]

(11)

Substituting these equations into Eq. (5), one has

\[ t_{\ell}(p, p'; s, P) = V_{\ell}(p, p') - (2\pi)^{-2} \int_{0}^{\pi} k^2 dk \frac{V_{\ell}(p, k) \times t_{\ell}(k, p'; s, P)}{k^2 - s - i\eta}. \]

(12)

This equation gives the full off-shell (nondiagonal) t-matrix, representing inelastic scattering. The on-shell (diagonal) t-matrix for elastic scattering is obtained simply by setting \( p = p' \) and \( s = P^2 \) (Bishop et al., 1977). This can be parametrized in terms of the phase-shifts \( \delta_{\ell} \) according to

\[ t_{\ell}(p, p; p^2) = -4\pi p^{-1} e^{i\delta_{\ell}(p)} \sin\delta_{\ell}(p); \]

(13)

or

\[ t_{\ell}(p; p; p^2) = -\frac{2\pi}{p} [\sin(2\delta_{\ell}(p)) + i(1 - \cos(2\delta_{\ell}(p))]]; \]

(14)

so that

\[ \tan(\delta_{\ell}(p)) = \frac{\text{Im} t_{\ell}(p; p; p^2)}{\text{Re} t_{\ell}(p; p; p^2)}. \]

(15)

\( \text{Im} t_{\ell}(p; p; p^2) \) and \( \text{Re} t_{\ell}(p; p; p^2) \) denote, respectively, the imaginary and real parts of \( t_{\ell}(p; p; p^2) \):
\[
\text{Re} \ t_\gamma(p; p^2) = -\frac{2\pi}{p} \sin(2\delta_\gamma(p));
\]
(16)

\[
\text{Im} \ t_\gamma(p; p^2) = -\frac{2\pi}{p} (1 - \cos(2\delta_\gamma(p))).
\]
(17)

The numerical details involved are not included here because they were discussed meticulously in previous work (Ghassib et al., 1976).

2.2.2 In the medium: GMF formalism

Such a theory is necessary for determining the medium phase shifts which, in turn, are needed for calculating \( B_q \). We have chosen here the GMF formalism because it incorporates in intermediate states hole-hole, as well as particle-particle, scattering. It is also valid for finite temperatures. In diagrammatic language, this takes into account only the ladder diagrams, which seem to be adequate in the low-density limit (Ghassib et al., 1976; Mosameh et al., 2014).

The central quantity in this formalism is the GMF T-matrix which in effect plays the role of a generalized scattering amplitude. It can be viewed as an effective interaction in momentum space, or as a Lippmann-Schwinger (LS) t-matrix, albeit ‘dressed up’ by the many-body medium. Extensive details regarding the technicalities involved, including the numerics, are well-explained in previous works (Ghassib et al., 1976; Bishop et al., 1977).

For a general many-fermion system, the T-matrix is given by

\[
T(p, p'; s, \tilde{p}) = u(|p - \tilde{p}'|) - (2\pi)^{-3} \int d\vec{k}u(|\vec{p} - \tilde{\vec{k}}|)
\]

\[
\times \left[ g_0(k, s)Q(k, \tilde{p}, \beta) - g_0^*(k, s)\tilde{Q}(k, \tilde{p}, \beta) \right] T(\tilde{k}, \tilde{p}', s, \tilde{\tilde{p}}).
\]
(18)

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The free two-body Green's function $g_0(k,s)$, representing propagation onto state $|\tilde{k}\rangle$, is defined by

$$g_0(k,s) \equiv \frac{1}{k^2 - s - i\eta};$$

(19)

its complex conjugate represents propagation out of this state, $\eta$ being a positive infinitesimal in the scattering region and zero otherwise.

The operator $Q$ ($\overline{Q}$) is the product of particle-particle (hole-hole) occupation probabilities. For Fermi systems, the (hole-)occupation probability is just the Fermi-Dirac distribution function $n(k)$, which represents the probability of a fermion having an energy $\varepsilon$; it is given by

$$n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon - \varepsilon_F)} + 1},$$

(20)

$\varepsilon_F$ being the Fermi energy.

$$n(\varepsilon) = \frac{1}{\frac{\beta h^2}{2m(k^2 - k_F^2)} + 1}.$$  

(21)

When subtracted from unity, the hole-occupation probability yields the particle-occupation probability. Thus,

$$Q(k,P,\beta) = \left(1 - n(k - P)\right)\left(1 - n(k + P)\right);$$

(22)

$$\overline{Q}(k,P,\beta) = n(k - P)\overline{n}(k + P).$$

(23)
The number density \( n = \frac{v k_p^3}{6\pi^2} \) (Walecka, 2004); so that \( k_p = \left(\frac{6\pi^2 n}{v}\right)^{1/3} \), where \( v \) is the spin-isospin degeneracy = 2 (4) for pure neutron matter (symmetric nuclear matter).

Repeating the above partial-wave decomposition scheme for the T-matrix, we have

\[
T_r(p, p'; s, P, \beta) = \frac{2m^*_sV_r(p, p')}{\hbar^2} - \int_0^{\infty} \frac{k^2 dk}{2\pi^2} \frac{2m^*_sV_r(p, k)f(k; P, \beta, s)T_r(k, p'; s, P, \beta)}, \tag{24}
\]

where

\[
f(k; P, \beta, s) \equiv \frac{Q(k, P, \beta)}{k^2 - s - i\eta} - \frac{\overline{Q}(k, P, \beta)}{k^2 - s + i\eta}.
\tag{25}
\]

In the absence of the many-body medium \((Q = 1, \overline{Q} = 0)\), i.e., in the free-scattering limit, Eq. (24) reduces to the LS equation (for the t-matrix).

The phase shifts \( \delta^E_r(p; P, \beta) \) can be determined by parametrizing the on-energy-shell T-matrix as follows (Ghassib et al., 1976; Obeidat et al., 2017):

\[
T_r(p, p; s, P, \beta) = T_r(p; P, \beta);
\]

\[
T_r(p; P, \beta) = -\frac{4\pi}{p} \exp\left[i\delta^E_r(p; P, \beta)\right] \sin\left[i\delta^E_r(p; P, \beta)\right] \frac{Q(p; P, \beta) + \overline{Q}(p; P, \beta)}{Q(p; P, \beta) + \overline{Q}(p; P, \beta)}; \tag{26}
\]

\[
T_r(p; P, \beta) = -\frac{2\pi}{p(Q(p; P, \beta) + \overline{Q}(p; P, \beta))} \left[\sin(2\delta^E_r(p; P, \beta)) + i(1 - \cos(2\delta^E_r(p; P, \beta)))\right]; \tag{27}
\]

so that
\[
\tan(\delta^E_\ell(p; P, \beta)) = \frac{\text{Im} T_\ell(p; P, \beta)}{\text{Re} T_\ell(p; P, \beta)}. \tag{28}
\]

\(\text{Im} T_\ell(p; P, \beta)\) and \(\text{Re} T_\ell(p; P, \beta)\) denote, respectively, the imaginary and real parts of \(T_\ell(p; P, \beta)\); they are defined by

\[
\text{Re} T_\ell(p; P, \beta) = -\frac{2\pi}{p(Q(p; P, \beta) + Q(p; P, \beta))}\sin\left(2\delta^E_\ell(p; P, \beta)\right) \tag{29}
\]

\[
\text{Im} T_\ell(p; P, \beta) = -\frac{2\pi}{p(Q(p; P, \beta) + Q(p; P, \beta))}\left(1 - \cos\left(2\delta^E_\ell(p; P, \beta)\right)\right) \tag{30}
\]

### 2.3 Quantum second virial coefficient

The second virial coefficient is a basic quantity that expresses the deviation of a gas from its ideal limit. It can be used for calculating other thermodynamic properties and for describing the phase diagram of the gas (McQuarrie, 1976; Feynman, 1992).

The phase shifts are related to the quantum second virial coefficient \(B_q\) through the Beth-Uhlenbeck formula (Uhlenbeck and Beth, 1936, Beth and Uhlenbeck, 1937; Lattimer and Ravenhall, 1978):

\[
B_q = +\frac{\lambda^3}{2^{5/2}} - 3\left[2^{3/2}\lambda^3 \sum_{E_n} \left(e^{-\beta E_n} - 1\right)\right] - \frac{2^{3/2}\lambda^5}{\pi^2} \int_0^\infty dk k^2 e^{iE_n(k)} e^{-\beta \varepsilon(k)}. \tag{31}
\]

Here: \(\lambda \equiv \left(2\pi\hbar^2 / mk_B T\right)^{1/2} = (2\pi/T)^{1/2}\) [in the natural system of units] is the thermal de Broglie wavelength, \(\beta = 1/T\), and \(\delta^E_\ell\) is the medium (or ‘effective’) \(\ell\)-partial phase shift corresponding to energy \(\varepsilon(k) = k^2/2\). The first term on the right-hand side is the ‘ideal-gas term’. The second represents the bound-state contribution; here there is only one bound state (in nuclear, but not neutron, matter) – namely, the deuteron (n+p) with binding energy \(E_B = -2.224\) MeV. The third term denotes the scattering-state
contribution. Both neutron (nn) and nuclear (np) matter are spin-$\frac{1}{2}$ Fermi systems. The above weighted sums are performed over even and odd values of \( \ell \) according to

\[
B_q = \frac{3}{4} B_{q,\ell=odd} + \frac{1}{4} B_{q,\ell=even}
\]

(32)

There are three interesting cases in this work. First, for our model potential, which

\[
B_q = + \frac{\lambda^3}{2^{5/2}} - \frac{2^{3/2} \lambda^5}{\pi^2} \int dk \delta_0^E(k) e^{-\beta E(k)}
\]

(33)

Second, the classical coefficient \( B_{cl}(T) \) (Mello et al., 1983; Pack, 1983; Hattig et al., 2003):

\[
B_{cl}(T) = 2\pi \int_0^\infty \left( 1 - e^{-\beta V(r)} \right) r^2 dr
\]

(34)

Third, the quantum correction \( B_{qc} \) to \( B_{cl} \) (Mello et al., 1983; Pack, 1983):

\[
B_{qc} = \left( \frac{\pi \hbar^2 \rho^3}{6m} \right) \int_0^\infty e^{-\beta V(r)} \left( \frac{dV}{dr} \right)^2 r^2 dr
\]

(35)

The comparison of quantum to classical results, guided by experimental values, should shed light on the ‘quantumness’ of our system.

2.4 Other thermophysical properties

The pressure \( P \) can be calculated from the virial equation of state (Feynman, 1992):

\[
\frac{P}{nk_B T} = 1 + nB
\]

(36)
The Helmholtz (free) energy is given by (Mamedov and Somuncu, 2017)

\[ F(T) \approx n k_B T \left( \ln (n \lambda^3) - 1 + B(T) n \right). \]

(37)

The entropy of the system can be evaluated from (Reif, 1965)

\[ S(T) = - \left( \frac{\partial F}{\partial T} \right). \]

(38)

The mean internal energy \( U \) is given by (Mamedov and Somuncu, 2017)

\[ U(T) \approx n k_B T \left( 1.5 - n T \frac{dB}{dT} \right). \]

(39)

The heat capacity is (Pathria and Beale, 2011)

\[ C_v(T) = \left( \frac{\partial U}{\partial T} \right). \]

(40)

Finally, the chemical potential \( \mu \) is given by (Mello et al., 1983)

\[ \mu(T) = k_B T \ln (n \lambda^3) + 2 n k_B T B(T). \]

(41)

3. Results and Discussion

3.1 Second virial coefficient, \( B \)

a) Zero- (or very low-) density limit

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$B_{cl}$, $B_{qcl}$, and $B_q$ are calculated in the T-range 5-60 MeV, in the zero- or very low-n limit. In this case, it is adequate to calculate $B_q$ in vacuum, using the LS t-matrix, which is a special case of the GMF T-matrix (with $Q = 1, \overline{Q} = 0$). The results are shown in Fig. 2. As expected, at high T (the upper limit of our temperature-range), the three coefficients approach each other asymptotically. In fact, $B_{cl}$ and $B_{qcl}$ coincide almost exactly, indicating that the quantum corrections are no longer necessary. Conversely, at lower T, $B_q$ becomes increasingly dominant; the system is a quantum fluid proper.

These results are applicable, in this n-limit, to both systems: pure neutron matter and symmetric nuclear matter.

Figure 2: The classical ($B_{cl}$), classical with quantum corrections ($B_{qcl}$), and quantum ($B_q$) second virial coefficients [F$^3$], as functions of temperature T [MeV], in the zero-density limit.

b) Higher densities
B_q is evaluated in the T-range 5-60 MeV at different values of n, for both pure neutron matter and symmetric nuclear matter. B_q is tangibly-dependent on n at low T; but at high T (≥ 25 MeV), it is only weakly-dependent on n and behaves like B_cl, as it should.

The results are displayed in Fig. 3 for pure neutron matter. Clearly, there exists a critical density n_c = 0.12 F^{-3}, such that at low n (< n_c), B_q is positive and decreases with increasing T. At low T (< 30 MeV), there exist two cases: For n < n_c, B_q is positive and increases with increasing n; the overall interaction is repulsive (Hurly and Moldover, 2000). For n ≥ n_c, B_q decreases; which means that the overall interaction is attractive. Figure 4 shows the results for B_q [F^3], using the GMF formalism, as functions of T [MeV], for symmetric nuclear matter. B_q has the same general behavior for different n. It is positive; which means that the overall interaction is repulsive, and increases with increasing n.

Figure 3: The quantum second virial coefficient B_q [F^3] at different densities n [F^{-3}], for pure neutron matter, as a function of temperature T [MeV], for both GMF and LS frameworks.
Figure 4: The quantum second virial coefficient $B_q(T) [F^3]$, as a function of temperature $T$ [MeV], at different number densities $n [F^3]$, for symmetric nuclear matter, using the GMF formalism.

3.2 Other thermophysical properties

a) Pure neutron matter

In Fig. 5, the pressure $P$ [MeV/F$^3$] is plotted versus $T$ for different $n [F^3]$. For $n < 0.12 F^3$, $P$ increases with increasing $T$. For $n \geq 0.12 F^3$, $P$ increases with increasing $T$ until it reaches a maximum at $T = T_i$ (inversion temperature), then decreases with increasing $T$. $T_i$ is a function of $n$. At $T = T_i$, $dP/dT = 0$; for $T < T_i$, $dP/dT > 0$; and for $T > T_i$, $dP/dT < 0$. The $P$-$T$ curve represents the equation of state of the system; it is a central result of this work.
Figure 5: The pressure $P \text{[MeV/F}^3\text{]}$ for pure neutron matter, as a function of temperature $T \text{[MeV]}$, at different number densities $n \text{[F}^3\text{]}$, using both GMF and LS frameworks.

The Helmholtz (free) energy per unit volume $F/\Omega \text{[MeV/F}^3\text{]}$ is plotted in Fig. 6 as a function of $T \text{[MeV]}$ at different $n \text{[F}^3\text{]}$. At low $n < 0.09 \text{ F}^3$, this energy decreases with increasing $T$. At $n = 0.09 \text{ F}^3$, $F$ behaves rather differently at low $T$; and for higher $n = 0.12 \text{ F}^3$, it attains a peak at $T = 20 \text{ MeV}$. 
Figure 6: The Helmholtz (free) energy per unit volume $F/\Omega$ [MeV/F$^3$] for pure neutron matter, as a function of temperature $T$ [MeV], at different number densities $n$ [F$^3$], using both GMF and LS frameworks.

Figure 7 displays the **entropy per unit volume** $S/\Omega$ [F$^3$]. For low $n < 0.09$ F$^3$, $S/\Omega$ increases with increasing $T$. For higher $n > 0.09$ F$^3$, there exists a critical temperature $T = T_i$: for $T < T_i$, $S/\Omega$ increases with increasing $T$ until $T = T_i$, then decreases with increasing $T$. 
Figure 7: The entropy per unit volume $S/\Omega \ [F^{-3}]$ for pure neutron matter, as a function of temperature $T \ [MeV]$, at different number densities $n \ [F^{-3}]$, using both GMF and LS frameworks.

Figures 8 and 10 show the mean internal energy per unit volume $U/\Omega \ [MeV/F^3]$ and the chemical potential $\mu \ [MeV]$, respectively, as functions of $T \ [MeV]$ at different $n \ [F^{-3}]$. For $n < 0.09 \ F^{-3}$, $U/\Omega$ decreases with increasing $T$ until $T = 10 \ MeV$; for $T > 10 \ MeV$, $U/\Omega$ increases. From Fig. 10, $\mu$ decreases for $n < 0.09 \ F^{-3}$. Thus, the behavior of $U/\Omega$ and $\mu$ at $n = 0.09 \ F^{-3}$ stands out from the rest of $n$-values – in conformity with our other results. This seems to indicate that this value of $n$ acts like a ‘critical density’, separating ‘low’ from ‘high’ $n$. 
Figure 8: The mean internal energy per unit volume $U/\Omega$ [MeV/F$^3$] for pure neutron matter, as a function of temperature $T$ [MeV], at different number densities $n$ [F$^{-3}$], using both GMF and LS frameworks.

In Fig. 9, the mean internal energy per volume $U/\Omega$ [MeV/F$^3$] is plotted versus $n$ [F$^{-3}$], at different $T$ [MeV]. This is another perspective for the equation of state of the system. Each curve is characterized by a cusp (at $n \sim 0.08$-0.09 F$^{-3}$), indicating a clear demarcation between two density regimes: low-$n$ and high-$n$ – which again is consistent with our other results.
Figure 9: The mean internal energy per volume $U/\Omega$ [MeV/F³] for pure neutron matter, as a function of density $n$ [F³], at different temperatures $T$ [MeV], using the GMF formalism.
Figure 10: The chemical potential $\mu$ [MeV] for pure neutron matter, as a function of temperature $T$ [MeV], at different number densities $n$ [F$^{-3}$], using both GMF and LS frameworks.

Figure 11 shows the **heat capacity per unit volume** $C_v/\Omega$ [F$^{-3}$], using both LS and GMF formalisms.
Figure 11: The heat capacity per unit volume $C_v/\Omega \ [F^{-3}]$ for pure neutron matter, as a function of temperature $T \ [MeV]$, at different number densities $n \ [F^{-3}]$, using both GMF and LS formalisms.

At this point it is worth comparing our results for pure neutron matter to the only other calculation that exists in the literature for the present model potential. This is the calculation carried out within the static fluctuation approximation (SFA) (Ghulam et al., 2008). The agreement is fair; in the sense that the properties concerned have the same general behavior, but the numbers differ. The discrepancy can be explained in that SFA is an independent-quasiparticle model; whereas GMF is an independent-quasipair model.

To illustrate: In the present work, at $T=15 \ MeV$ and $n = 0.12 \ F^{-3}$, $P$ is $\sim 5.34 \ MeV/F^3$; at $T = 45 \ MeV$ for the same density, $P$ is $\sim 7.59 \ MeV/F^3$. In SFA, at the same density, $P$ is $\sim 4.2 \ MeV/F^3$ at $T = 15 \ MeV$ and $\sim 6.1 \ MeV/F^3$ at $T = 45 \ MeV$. Next, at $n = 0.12 \ F^{-3}$ and $T= 50 \ MeV$, $S/\Omega$ is $\sim 0.21$ (GMF), and $\sim 0.05$ (SFA). Finally, in GMF: at $n$
= 0.12 F\(^{-3}\) and T= 20 MeV, U/\(\Omega\) is \(\sim 4.7 \text{ MeV/F}^3\), and for the same n at T= 50 MeV, it is \(\sim 10.7 \text{ MeV/F}^3\). In SFA, the corresponding values are \(\sim 9 \text{ MeV/F}^3\) and 20 MeV/F\(^3\), respectively. Further, at n = 0.12 F\(^{-3}\) and T= 20 MeV, C\(_v\)/\(\Omega\) is \(\sim 0.209 \text{ F}^{-3}\) (GMF), and \(\sim 0.2 \text{ F}^{-3}\) (SFA).

These results are indicative of the validity of our model potential as a first, preliminary attempt to debug our codes and for orientation purposes. After all, as stated earlier, this model potential retains the overall features of the NN potential.

b) Symmetric nuclear matter

For this system, the pressure P [MeV/F\(^3\)] is shown in Fig. 12, as a function of T [MeV], at different n [F\(^{-3}\)]. At low T (\(\leq 20\) MeV), P decreases; but at higher T (\(\geq 20\) MeV), P increases. It is worth recalling that the equilibrium density of nuclear matter is 0.17 nucleons/F\(^3\) (Bethe, 1971; Lattimer and Prakash, 2000). However, the present calculation pertains to a model potential; so that the equilibrium density is necessarily different. Once again, the P-T curve represents the equation of state of this system.

Another useful relation is P [MeV/F\(^3\)] versus n [F\(^{-3}\)] at different [MeV]. This is displayed in Fig. 13. Interestingly, the cusp pointed out earlier (Fig. 9), separating low-n and high-n regimes, makes another appearance at \(\sim 0.08 \text{ F}^{-3}\).
Figure 12: The pressure $P$ [MeV/F$^3$] for symmetric nuclear matter, as a function of density $n$ [F$^{-3}$], at different temperatures $T$ [MeV], within the GMF framework.
Figure 13: The pressure $P$ [MeV/F$^3$] for symmetric nuclear matter, as a function of density $n$ [F$^{-3}$], at different temperatures $T$ [MeV], within the GMF framework.

Figure 14 shows the **Helmholtz (free) energy per nucleon** $F/N$ [MeV] as a function of $T$, for various $n$. Here again, $F$ increases monotonically with increasing $n$ and decreasing $T$. However, at high $T$, $F$ continues to decrease well beyond our $T$-range.

![Figure 14: The Helmholtz (free) energy per nucleon $F/N$ [MeV] for symmetric nuclear matter, as a function of temperature $T$ [MeV], at different number densities $n$ [F$^{-3}$], using the GMF framework.](image)

The entropy per nucleon $S/N$ [which is dimensionless in our natural system of units] is given in Fig. 15 as a function of $T$ [MeV], for various $n$ [F$^{-3}$]. Clearly, the behavior of $S/N$ is similar to that of $P$. For $n < 0.08$ F$^{-3}$, $S/N$ increases with increasing $T$. For $n \geq 0.08$ F$^{-3}$, $S/N$ decreases with increasing $T$ until it reaches a maximum at $T = 25$ MeV (inversion temperature), then increases with increasing $T$. 
Figure 15: The entropy per nucleon $S/N$ for symmetric nuclear matter, as a function of temperature $T$ [MeV], at different number densities $n$ [$F^{-3}$], using the GMF framework.

Figures 16 and 17 display the mean internal energy per nucleon $U/N$ [MeV] first, as a function of $T$ [MeV], at different $n$ [$F^{-3}$]; and, second, as a function of $n$ [$F^{-3}$], at different $T$ [MeV]. In Fig. 17, the ‘inverted cusp’ can be viewed as a dip; i.e., $U$ exhibits a ‘minimum’ with $n$, indicating equilibrium. Another point worth commenting on is the fact that, from Fig. 16, following the dip, $U$ increases linearly with $T$. This is consistent with the classical theorem of equipartition of energy; the nucleons behave then like free particles. Likewise, from Fig. 17, following the ‘dip’, $U \sim n$. 
Figure 16: The mean internal energy per nucleon $U/N$ [MeV] for symmetric nuclear matter, as a function of temperature $T$ [MeV], at different number densities $n$ [$F^{-3}$], using the GMF framework.
Figure 17: The mean internal energy per nucleon $U$ [MeV/N] for symmetric nuclear matter, as a function of density $n$ [F$^{-3}$], at different temperatures $T$ [MeV], using the GMF formalism.

The chemical potential $\mu$ [MeV] is displayed in Fig. 18 as a function of $T$ [MeV], for various $n$ [F$^{-3}$]. The behavior of $\mu$ is reminiscent of that of $F$. This follows from basic thermodynamics.
Figure 18: The chemical potential $\mu$ [MeV] for symmetric nuclear matter, as a function of temperature $T$ [MeV], at different number densities $n$ [$F^{-3}$], using the GMF formalism.

The **heat capacity per nucleon** $C_v/N$ [which, again, is dimensionless in our natural system of units] is plotted in Fig. 19 as a function of $T$ [MeV] for different $n$ [$F^{-3}$].
4. Conclusion

In this work, a new approach was proposed for determining the equation of state for nonrelativistic nucleonic matter (in the regime where the system behaves like a nonideal gas). The starting point was an accurate calculation of the quantum second virial coefficient $B_q$ of the system. To this end, the Beth-Uhlenbeck formula was invoked. This necessitated the computation of the medium phase shifts which, in turn, were evaluated using the Galitskii-Migdal-Feynman formalism. The integral equations involved were solved using a highly-accurate matrix-inversion method. Having obtained $B_q$, one could easily find the equation of state, and other thermophysical properties, of the system, using standard expressions.

The extensive results of this work indicate that the present approach is workable. This motivates us to go beyond the simple model adopted here to a realistic NN potential;
for example, the Reid 93 soft core potential. Work is already in progress for this potential and preliminary results are very encouraging. Comparison of our results with those of other previous work, whenever possible, indicated fair agreement.

5. Acknowledgments

One of us (H. B. Ghassib) wishes to thank The University of Jordan for granting him a sabbatical leave in the academic year 2017-18, during which this work was undertaken. We are grateful to Dr. Ali Al-Harazneh for his substantive help in setting up a preliminary version of our computer codes.

6. References


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© 2018 British Journals ISSN 2047-3745


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